Quantum Algorithms for Systems of Linear Equations

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Workshop at the Intersection of Machine Learning and Quantum Information
University of Maryland
September 2018

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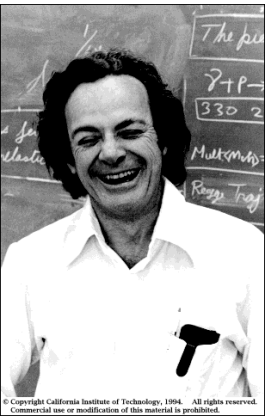
References:

- “Quantum linear systems algorithm with exponentially improved dependence on precision”, A.M. Childs, R. Kothari, and **R.D. Somma**, SIAM J. Comp. **46**, 1920 (2017).

- “Quantum algorithms for linear systems inspired by adiabatic quantum computing”, Y. Subasi, **R.D. Somma**, and D. Orsucci, arXiv:1805.10549 (2018).

A brief history of results in quantum computing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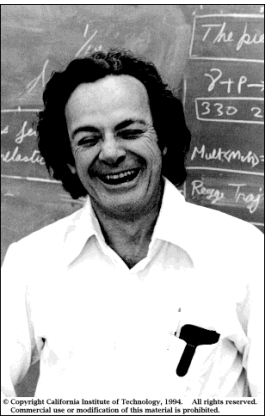
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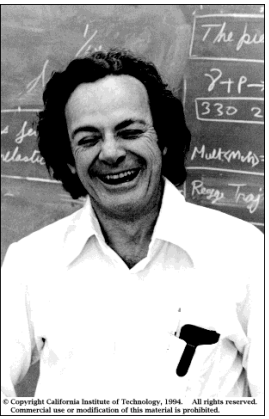
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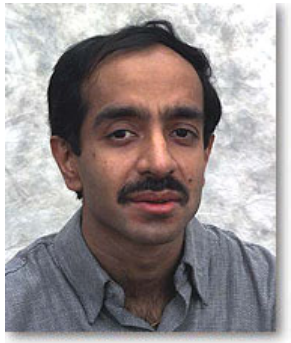

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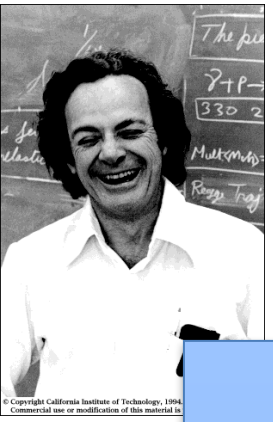

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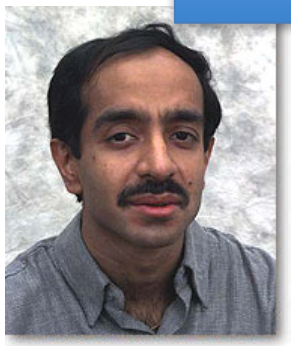
Other quantum algorithms for linear algebra problems?



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Linear Equations: An important problem

Let's consider the problem of solving a system of linear equations or the related problem of inverting a matrix:

$$A.\vec{x} = \vec{b} \quad \longrightarrow \quad \vec{x} = A^{-1}\vec{b}$$

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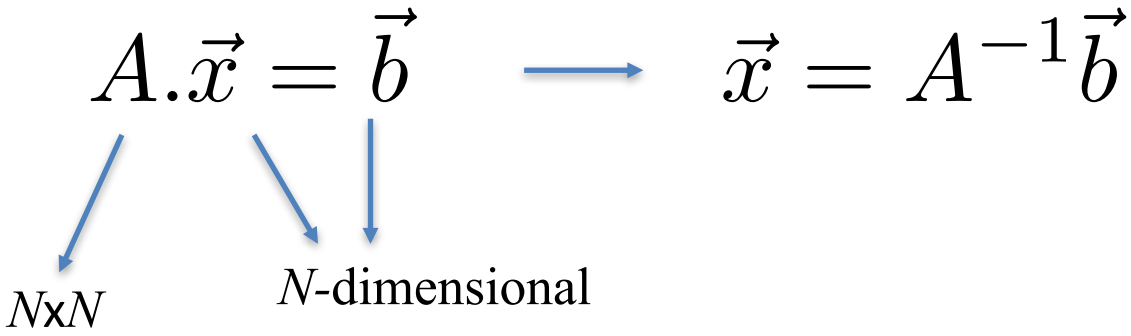

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- Note: This is a somewhat different problem (QLSP) and classical algorithms may do better in this case. However, the QLSP is BQP-Complete.

Quantum Linear System Problem (QLSP)

$$A.\vec{x} = \vec{b}$$

Assumptions

- A is Hermitian of dimension NxN
- A is s-sparse
- A is invertible and its condition number is $\kappa<\infty$
- The spectral norm of A is bounded by 1

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Proportional to $\vec{b}$

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Let U_A be a procedure that runs in time T_A and computes the entries of A

Let U_b be a procedure that runs in time T_b and prepares the state $|b\rangle$

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Let $C_A(t,\epsilon)$ be the cost of simulating e^{-iAt} with precision ϵ

Let U_b be a procedure that runs in time T_b and prepares the state $|b\rangle$

The goal is to prepare a quantum state $|\tilde{x}\rangle$ such that $\||\tilde{x}\rangle - |x\rangle\| \leq \epsilon$ with probability $\geq 1/2$, with a flag indicating success

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The goal is to prepare a mixed state ρ_x such that $\frac{1}{2}\mathrm{Tr}\,|\rho_x - |x\rangle\langle x|\,| \leq \epsilon$

Hamiltonian simulation

Note: Recent advances in Hamiltonian simulation resulted in

$$C_A(t,\epsilon) = \tilde{O}(tsT_A \log(t/\epsilon))$$

- Complexity almost linear in the evolution time
- Complexity is polylogarithmic in the inverse of a precision parameter

D. Berry, A. Childs, R. Cleve, R. Kothari, and RDS, PRL 114, 090502 (2015)
D. Berry, A. Childs, and R. Kothari, FOCS 2015, 792 (2015)
G.H. Low and I. Chuang, PRL 118, 010501 (2017)

Quantum Linear System Problem (QLSP)

Some applications:

- In physics, where the goal is to compute the expectation value of the inverse of a matrix. This idea was used in [1] for obtaining the resistance of a network.

- In stat mech, where, e.g., estimating the hitting time of a Markov chain also reduces to computing the expectation value of the inverse of a matrix [2]

- In ML, for solving problems related to least-squares estimation [3], by applying the pseudoinverse: $\arg\min_{\lambda\in\mathbb{C}^M} \|F\vec{\lambda} - \vec{y}\| \to \lambda = \frac{1}{F^\dagger F} F^\dagger \vec{y}\,.$

- For solving certain linear differential equations [4]: $\dot{\vec{x}}(t) = A(t)\vec{x}(t) + \vec{b}(t)$

[1] G. Wang, arXiv:1311.1851 (2013). [2] A. Chowdhury and R. Somma, QIC 17, 0041 (2017)
[3] N. Wiebe, D. Braun, and S. Lloyd, PRL 109, 050505 (2012).
[4] D. Berry, A. Childs, A. Ostrander, and G. Wang, CMP 356, 1057 (2017)

Quantum Linear System Problem (QLSP)

A note: Even for those applications, a number of assumptions must be made in order to obtain quantum speedups. These assumptions include efficient preparation of certain states (of exp many amplitudes), nice scaling of the condition number, and solving certain problems like computing expectation values. For these reasons, shown quantum speedups are typically polynomial.

The HHL Algorithm for the QLSP [5]

[HHL08] There exists a quantum algorithm that solves the QLSP with complexity

$$\tilde{O}\left[\kappa(T_b + C_A(\kappa/\epsilon, \epsilon/\kappa))\right]$$

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$$\tilde{O}\left[\kappa T_b + C_A(\kappa/\epsilon^3, \epsilon)\right] \longrightarrow \text{Almost linear in } \kappa!$$

- Note that the best Hamiltonian simulation methods have query and gate complexities almost linear in evolution time and logarithmic in precision

[5] Harrow, Hassidim, Lloyd, PRL 103, 150502 (2009)
[6] A. Ambainis, STACS 14, 636 (2012)

A quick view of the HHL algorithm and VTAA

HHL has three registers: I that holds the state $|b\rangle$, E that holds an estimate of the eigenvalue, and O that is an ancilla qubit. $\mathcal{H} = \mathcal{H}_I \otimes \mathcal{H}_E \otimes \mathcal{H}_O$

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$$|b\rangle \rightarrow \sum_{j=0}^{N-1} c_j|v_j\rangle_I|\tilde{\lambda}_j\rangle_E$$

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This register contains the eigenvalue estimate (superposition):

- It suffices to have the estimate with relative precision ϵ
- Order $\log(\kappa/\epsilon)$ ancillary qubits

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Amplitude amplification for amplifying the amplitude of the $|0\rangle_O$ state

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The action of 1/ (κ A) on the state reduces its amplitude by order 1/ κ and order κ amplitude amplification rounds are needed

From here we see that we need to evolve with A for time that is, at least, order κ^2/ϵ

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- First we do a bad-precision phase estimation to distinguish large from small eigenvalues. This may be done evolving with A for time independent of κ
- Then we implement a rough approximation of $1/\kappa A$ to eigenstates of large eigenvalue
- We need order κ amplitude amplification steps
- We implement an accurate approximation of $1/\kappa A$ to eigenstates of small eigenvalue
- Amplitude amplification for order 1 steps
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The complexity of VTAA in terms of precision is worse than that of HHL

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This talk: two quantum algorithms for the QSLP

- I will present two quantum algorithms for the QLSP that improve previous results in different ways:

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- This results in an exponential improvement on the precision parameter
- It can be improved using a version of VTAA to:

[7] There exists a quantum algorithm that solves the QLSP with complexity

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[7] A. Childs, R. Kothari, **RDS**, SIAM J. Comp. **46**, 1920 (2017).

This talk: two quantum algorithms for the QSLP

Why these improvements are important?

- The previous result allowed us to prove a polynomial quantum speedup for hitting time estimation in terms of the spectral gap of a Markov chain and precision (A. Chowdhury, R.D. Somma, QIC 17, 0041 (2017)).

- Having a small complexity dependence on precision is important for, e.g., computing expectation values of observables at the quantum metrology limit.

This talk: two quantum algorithms for the QSLP

- I will present two quantum algorithms for the QLSP that improve previous results in different ways:

[8] There exists a quantum algorithm that solves the QLSP by evolving with Hamiltonians that are linear combinations of (products of) A, the projector in the initial state, and Pauli matrices. The overall evolution time is $\tilde{O}(\kappa/\epsilon)$

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Using Hamiltonian simulation, this transfers to complexity $\tilde{O}(\kappa T_b/\epsilon + C_A(\kappa/\epsilon, \epsilon))$

- The method is very different and based on adiabatic evolutions. It does not require of complicated subroutines such as phase estimation and variable time amplitude amplification, therefore reducing the number of ancillary qubits substantially.

[8] Y. Subasi, **RDS**, D. Orsucci, arXiv:1805.10549 (2018).

This talk: two quantum algorithms for the QSLP

Why this improvement is important?

- Phase estimation and VTAA require several ancillary qubits (beyond those needed for Hamiltonian simulation)
- Within two weeks of posting our result, a group implemented our algorithm in NMR, claiming that it is the largest simulated instance so far (8x8) [9]

[9] J. Wen, et.al., arXiv:1806.0329 (2018)

First algorithm: A Fourier approach for solving the QSLP

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Recall that the main goal is achieved by implementing $1/A$ to the initial state $|b\rangle$

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- $1/A$ is not unitary and we need to find a unitary implementation for it. We then go through a sequence of approximations:

$$\frac{1}{x} = \int_0^\infty dy\; xye^{-(xy)^2/2}\;,\;\; xye^{-(xy)^2/2} = \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dz\; ze^{-z^2/2} e^{-ixyz}$$

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$$\frac{1}{A} \approx_\varepsilon \frac{i\delta y\, \delta z}{\sqrt{2\pi}} \sum_{j=0}^{J} \sum_{k=-K}^{K} z_k\, e^{-(z_k)^2/2}\, e^{-iA(y_j z_k)}$$

we are getting closer: Linear combination of unitaries

First algorithm: A Fourier approach for solving the QSLP

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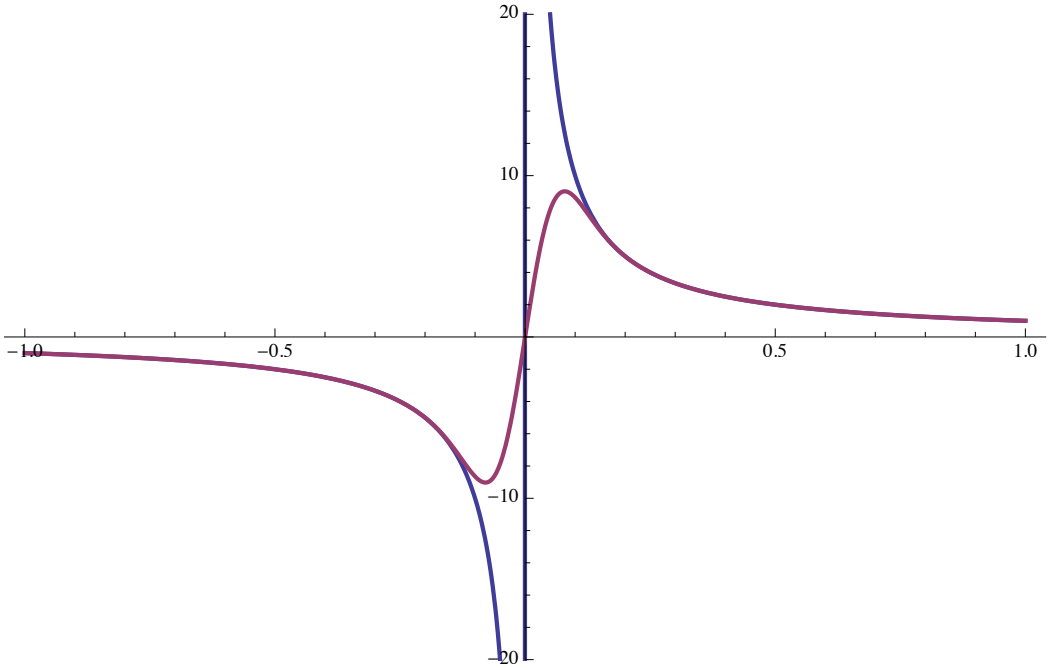

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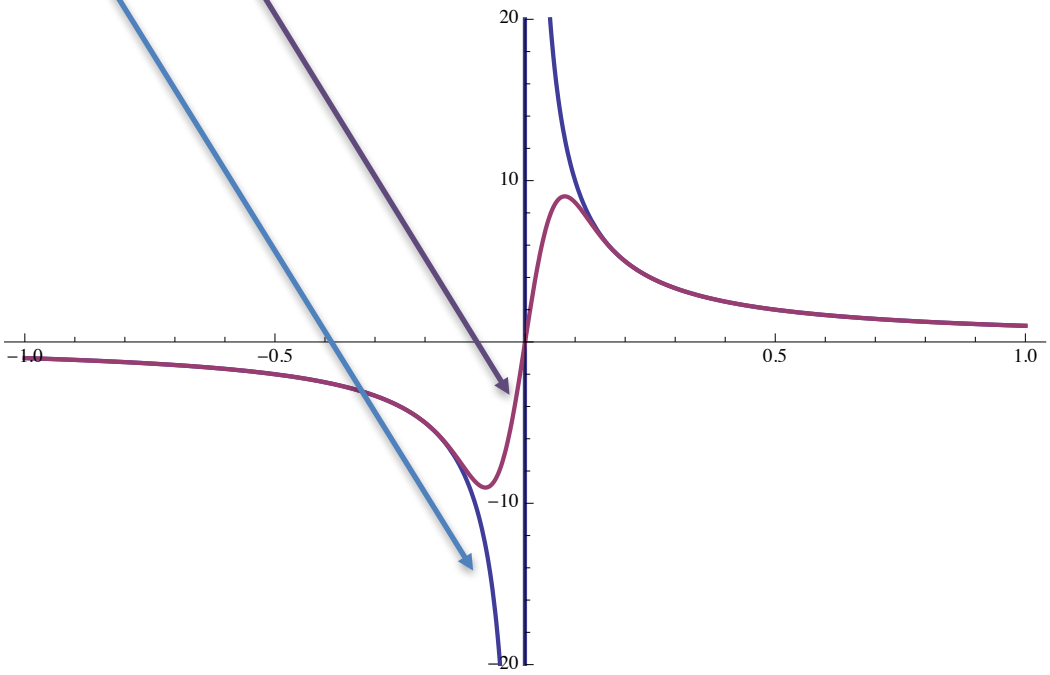

The accuracy of the approximation will determine $J, K, \Delta y, \Delta z$

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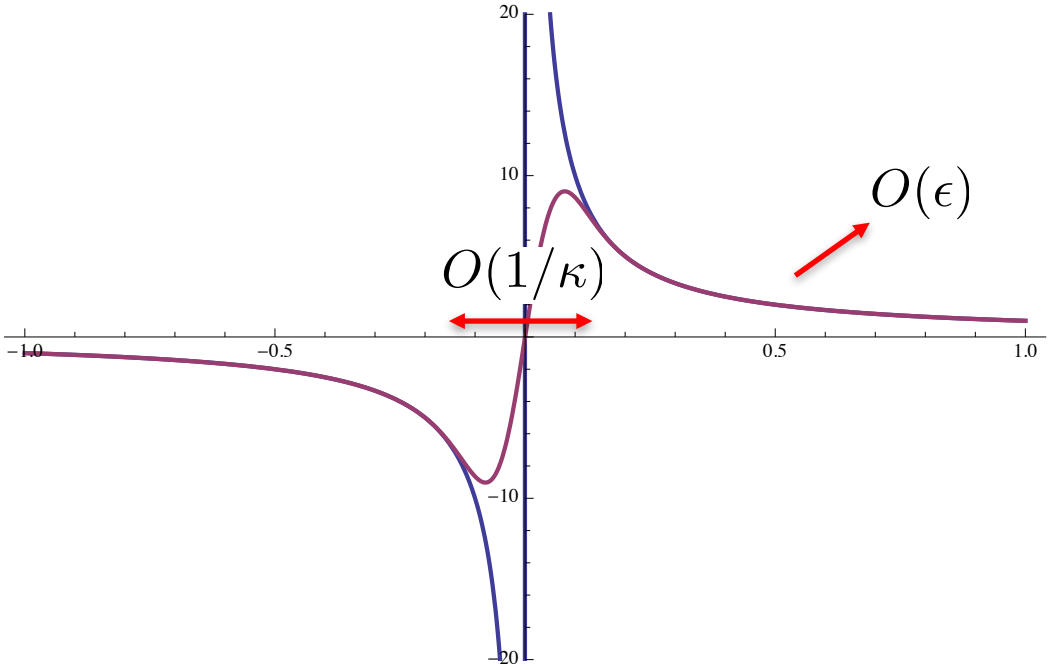

$$J = \tilde{O}(\kappa/\epsilon)$$
$$K = \tilde{O}(\kappa)$$
$$\Delta y = \tilde{\Omega}(\epsilon)$$
$$\Delta z = \tilde{\Omega}(1/\kappa)$$

The maximum "evolution time" under A in the approximation of $1/A$ is

$$(J\Delta y)(K\Delta z) = O(\kappa \log(\kappa/\epsilon))$$

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- So far we approximated $1/A$, within the desired accuracy, by a finite linear combination of unitaries. Each unitary corresponds to evolving with A for certain time, and the max evolution time is almost linear in the condition number

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QLSP  Hamiltonian simulation

Implementing a linear combination of unitaries

Suppose we want to implement the operator $\lambda_1 U_1 + \lambda_2 U_2$ to some state $|\psi\rangle$ where $\lambda_i \geq 0$, $\lambda_1 + \lambda_2 = 1$, and U_i unitary

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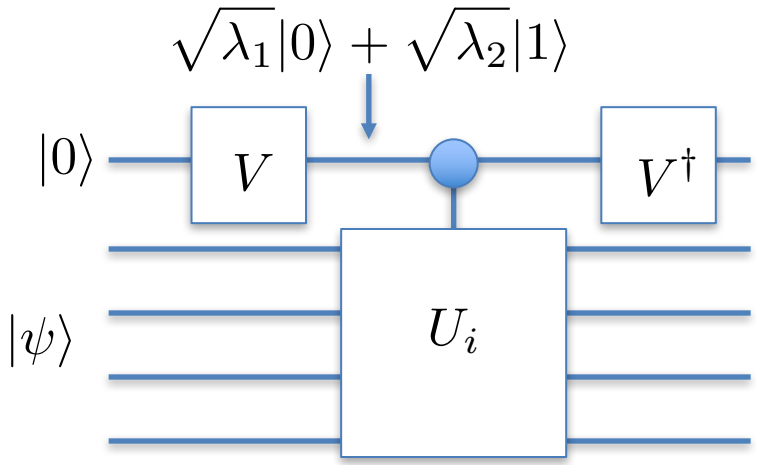

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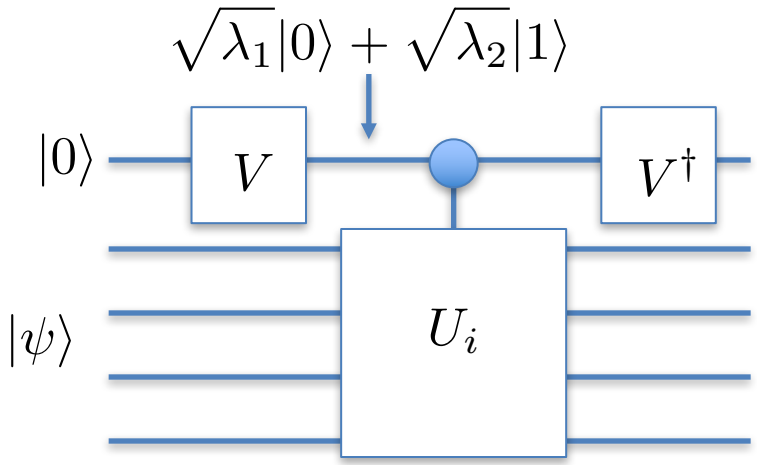

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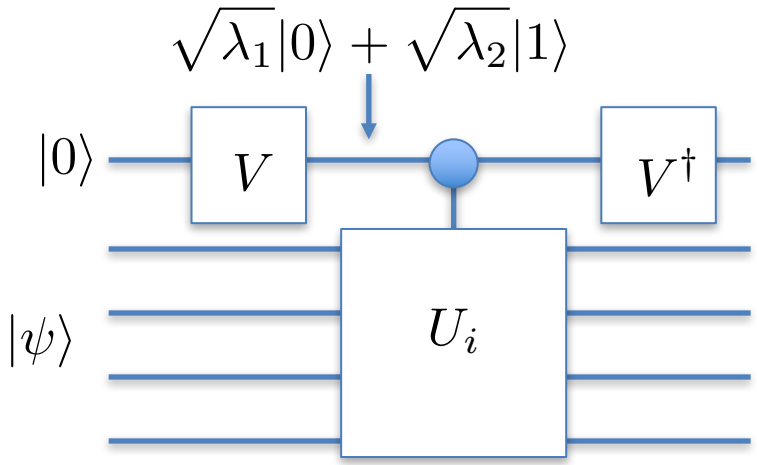

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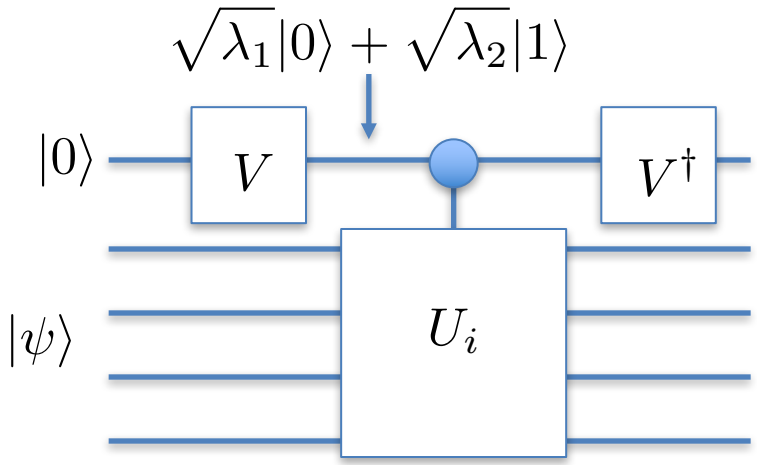

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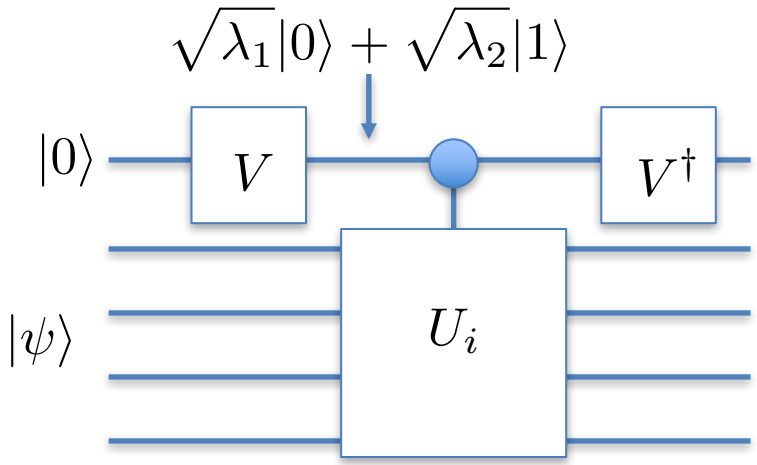

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Amplitude amplification to obtain the correct part

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This is almost quadratic in the condition number. To improve it to almost linear we use a version of VTAA that doesn't ruin the logarithmic scaling in precision

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Using the best Hamiltonian simulation methods, this is almost linear in the condition number and polylog in inverse of precision

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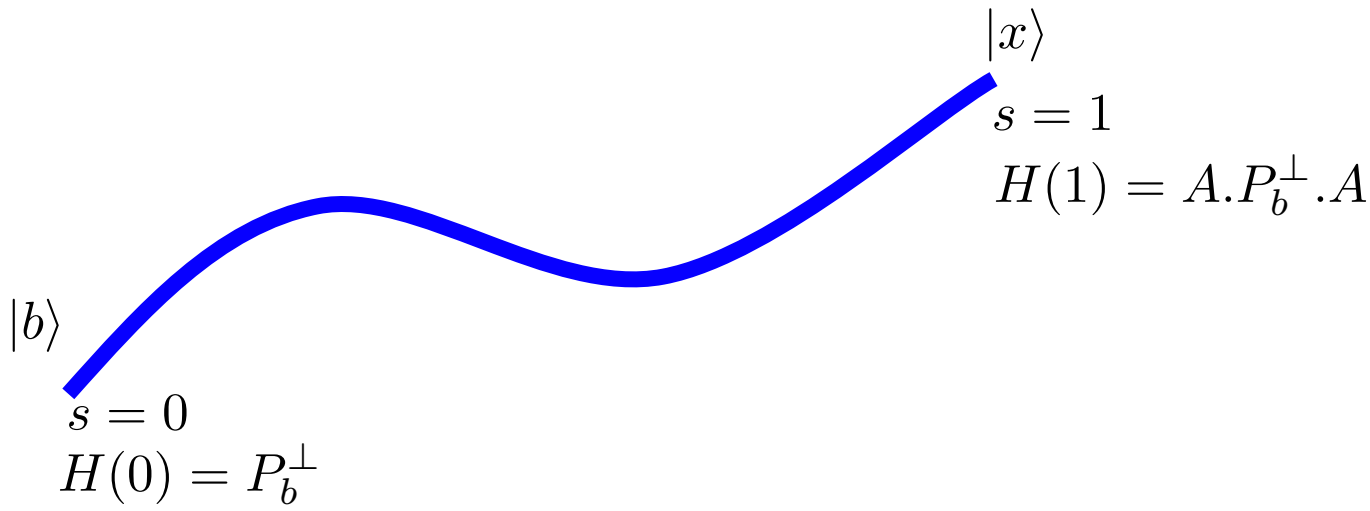

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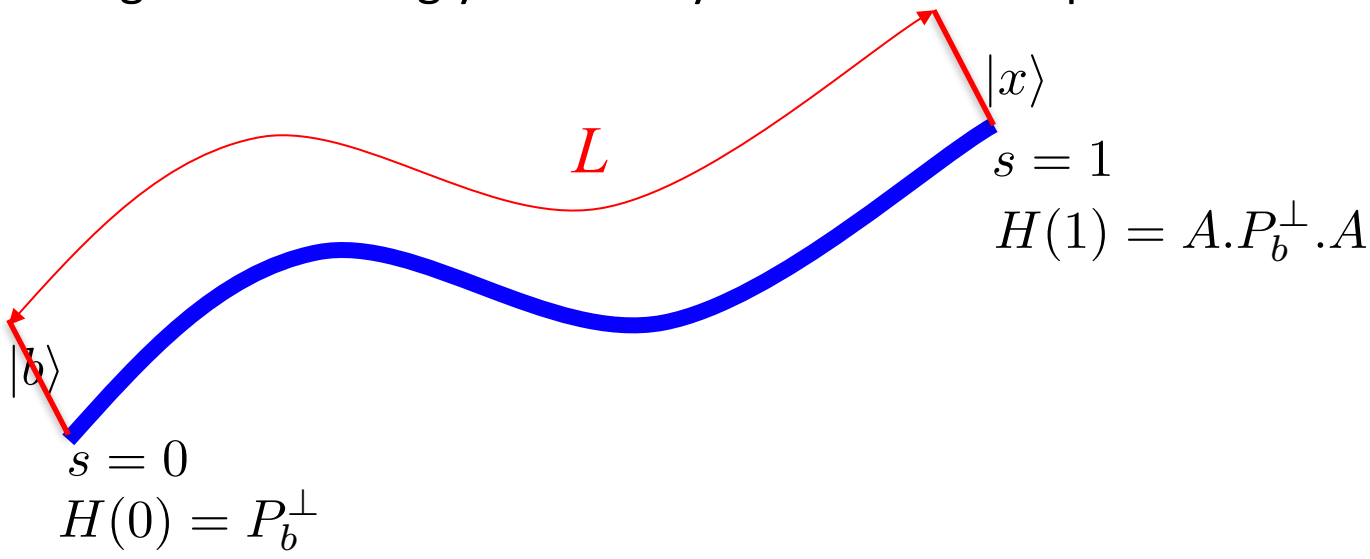
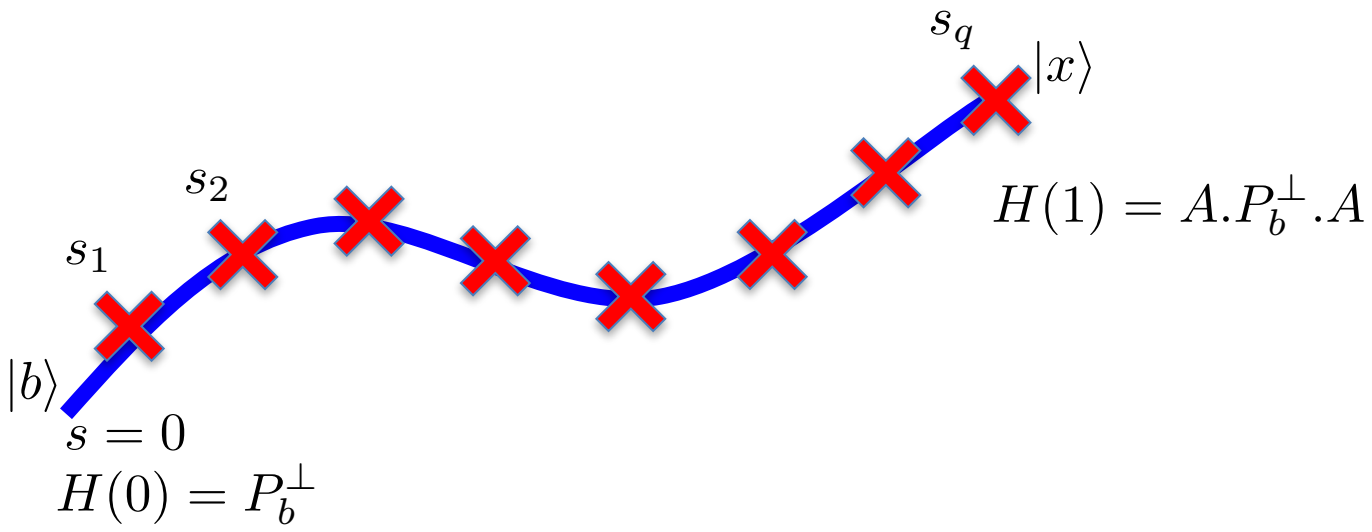


- The minimum eigenvalue gap is order 1/ κ^2 and the length of the path L is log(κ)

The randomization method to prepare eigenstates []



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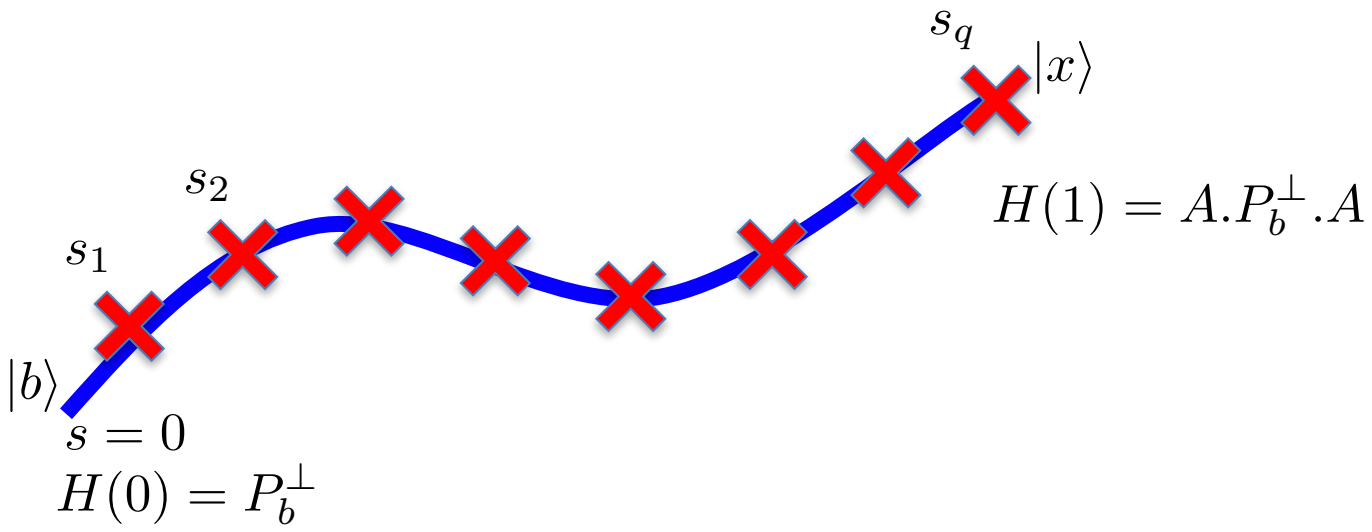

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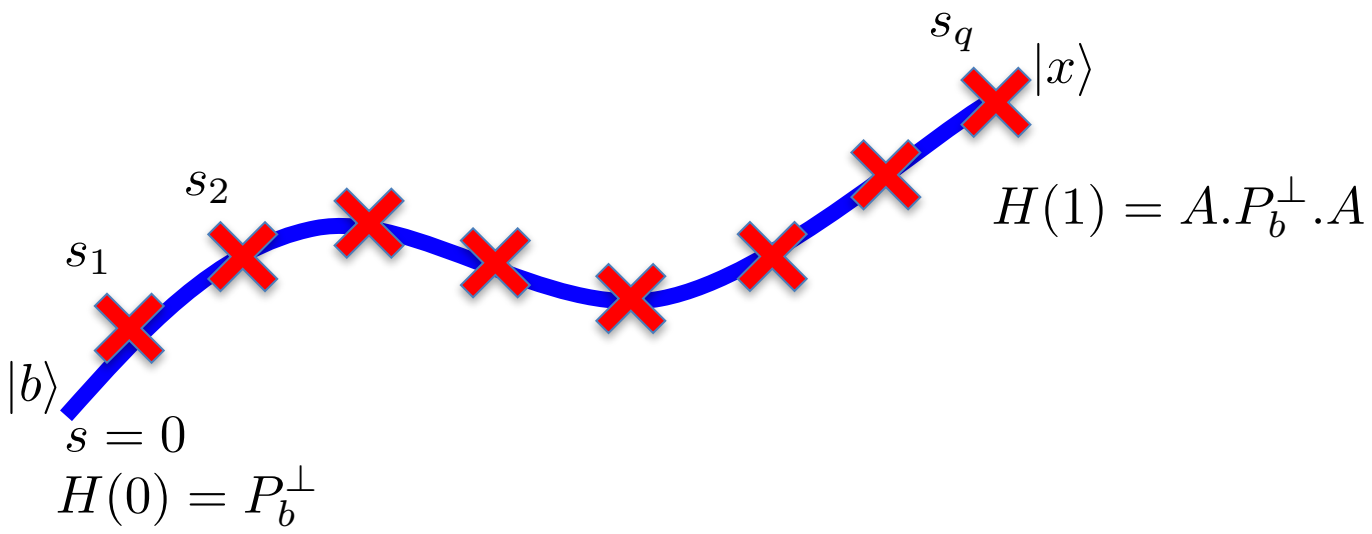

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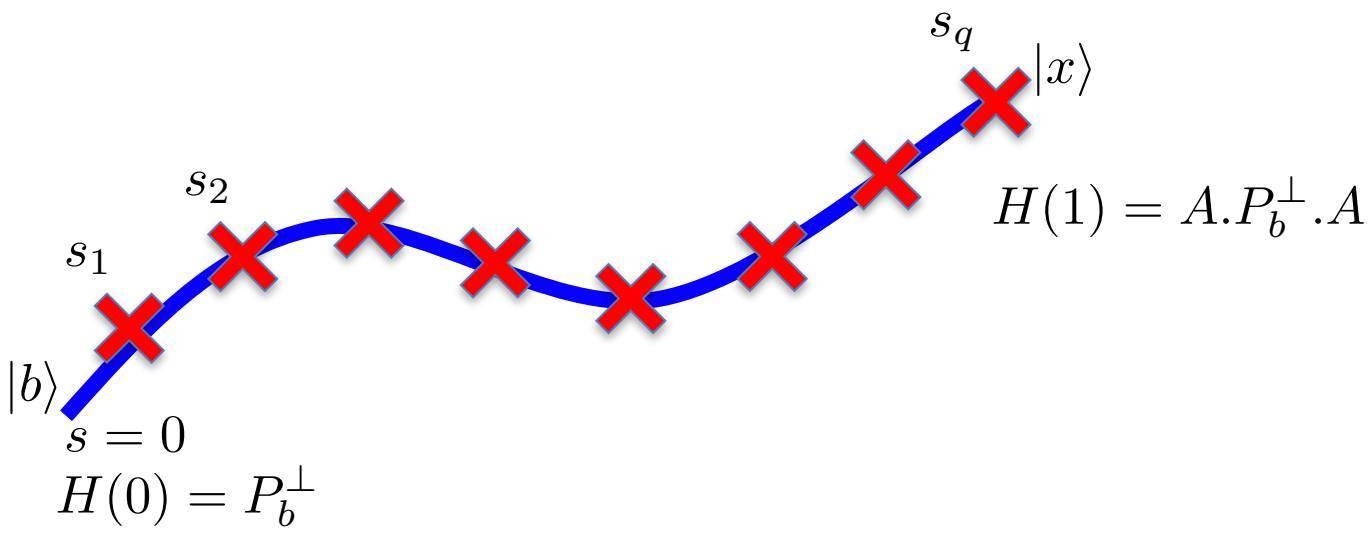

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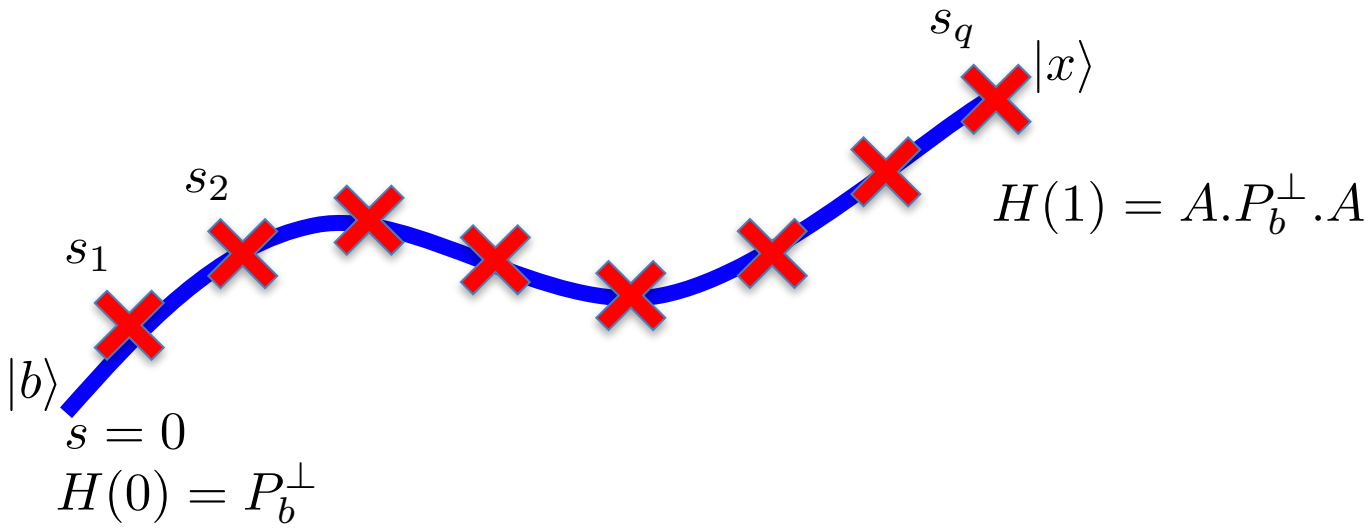
L is the path length
Δ is the min gap
ϵ is the error (trace norm)

The randomization method for the QLSP []



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Let $|x(s)\rangle$ be the eigenstate of 0-eigenvalue of $H(s)$.
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- Note that the path length did not change. The only change for the RM is the use of a different Hamiltonian.

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- The case of non-positive matrix A can be analyzed similarly using

$$A(s) = (1-s)(\sigma_z^{anc} \otimes I) + s(\sigma_x^{anc} \otimes A)$$

The randomization method, the QLSP, and the gate model

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- The complexity in terms of queries for A is almost order $C_A(\kappa/\epsilon, \epsilon)$
- The scaling in the precision parameter can be done polyligarithmic by using faster methods for eigenpath traversal [11]

[11] S. Boixo, E. Knill, and R.D. Somma, arXiv:1005.3034 (2010)

Some conclusions and observations

- Quantum computing seems promising. Several quantum algorithms for problems in linear algebra with significant speedups exist

- I presented quantum algorithms to solve the quantum linear systems problem. The techniques can be generalized to apply other operators (other than the inverse of a matrix) to quantum states.

- The advantages of the first algorithm are in that the complexity dependence on precision is only polylogarithmic, exponentially improving previous algorithms for this problem

- The advantages of the second algorithm are in that it doesn't require many ancillary qubits and the problem reduces to a simple Hamiltonian simulation problem

- It would be important to understand the applicability of this algorithm to scientific problems beyond the ones I mentioned. How important are the problems and algorithms?